

THE DERIVATION OF TENSORS FROM TENSOR FUNCTIONS

A Dissertation

SUBMITTED TO THE BOARD OF UNIVERSITY STUDIES OF THE JOHNS HOPKINS
UNIVERSITY IN PARTIAL FULFILMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

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THE DERIVATION OF TENSORS FROM TENSOR FUNCTIONS.*

By ALFRED K. MITCHELL.

Introduction. E. Schroedinger † has given a rule for deriving a tensor from an invariant tensor function. In the first part of the present paper ‡ a proof is given of the theorem that if Φ is any invariant function of a tensor its derivative with respect to a component of this tensor is itself a component of a tensor. It is also proved that the derivative of a tensor function of a tensor produces a tensor of higher rank. The first of these theorems is then applied to the invariants of a mixed tensor of rank 2, and the invariants of the tensor so derived are considered. In this way it is seen that if F_s^r is any polynomial function $F = a_0 + a_1 E + a_2 E^2 + \dots$ of the matrix E , then its invariants can be expressed in terms of the invariants of E and hence the derivatives of the invariants of F are functions of the derivatives of the invariants of E .

1. By a space of n dimensions we mean a continuous arrangement of points; a point being a set of n ordered real numbers $(x^1, x^2, x^3, \dots, x^n)$.

A set of n equations

$$\bar{x}^r = \bar{x}^r(x^1, x^2, \dots, x^n) \quad (r = 1, 2, \dots, n)$$

in which the functions $\bar{x}^r$ are single valued for all points and which can be solved § so as to yield a set of n equations

$$x^r = x^r(\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n)$$

in which the functions x^r are single valued, define a transformation of coördinates in the n dimensional space.

A set of $n^{(p+q)}$ functions $X_{s_1 s_2 \dots s_q}^{r_1 r_2 \dots r_p}$, defined with respect to a coördinate system (x) , and from which we obtain, in any other coördinate system $(\bar{x})$, the corresponding $n^{(p+q)}$ functions $\bar{X}_{\bar{s}_1 \bar{s}_2 \dots \bar{s}_q}^{\bar{r}_1 \bar{r}_2 \dots \bar{r}_p}$, by means of the $n^{(p+q)}$ equations of transformation ¶

* Presented to the American Mathematical Society, October 26, 1929.

† See *Annalen der Physik*, Vol. 82 (1927), p. 265.

‡ The author is indebted to Professor F. D. Murnaghan for helpful suggestions concerning the material of this paper.

§ This implies that the Jacobian determinant $\partial(\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n)/\partial(x^1, x^2, \dots, x^n)$ is not identically zero. See Goursat-Hedrick, *Mathematical Analysis*, Vol. I, Chap. 2.

¶ In these equations and throughout this paper Greek letters occurring as indices in pairs will be used as summation or umbral labels, the summation from 1 to n . See O. Veblen, *Cambridge Tracts*, No. 24, Chaps. 1 and 2.

$$\bar{X}_{s_1 s_2 \dots s_q}^{r_1 r_2 \dots r_p} = X_{\beta_1 \beta_2 \dots \beta_q}^{\alpha_1 \alpha_2 \dots \alpha_p} \frac{\partial \bar{x}^{r_1}}{\partial x^{\alpha_1}} \frac{\partial \bar{x}^{r_2}}{\partial x^{\alpha_2}} \dots \frac{\partial \bar{x}^{r_p}}{\partial x^{\alpha_p}} \frac{\partial x^{\beta_1}}{\partial \bar{x}^{s_1}} \frac{\partial x^{\beta_2}}{\partial \bar{x}^{s_2}} \dots \frac{\partial x^{\beta_q}}{\partial \bar{x}^{s_q}}$$

is said to define a mixed tensor of rank $p + q$ which is contravariant of rank p and covariant of rank q .

Any function X of the coördinates whose value is unchanged when we change the coördinate system is called an invariant or scalar function. The equation of transformation is

$$\bar{X} = X.$$

According to the rule of composition of tensors* we can form an invariant from a contravariant tensor of rank p , $X^{r_1 r_2 \dots r_p}$ and a covariant tensor of rank p , $X_{r_1 r_2 \dots r_p}$ by forming their scalar product

$$X^{\alpha_1 \alpha_2 \dots \alpha_p} X_{\alpha_1 \alpha_2 \dots \alpha_p} (\alpha_1, \alpha_2 \dots \alpha_p \text{ summation labels}).$$

The converse of the rule of composition of tensors may be stated for the general case as follows. If the functions $X_{b_1 \dots b_q s_1 \dots s_m}^{\alpha_1 \dots \alpha_p r_1 \dots r_l}$ have such a law of transformation that the summation

$$X_{\beta_1 \dots \beta_q s_1 \dots s_m}^{\alpha_1 \dots \alpha_p r_1 \dots r_l} Y_{\alpha_1 \dots \alpha_p}^{\beta_1 \dots \beta_q}$$

is a mixed tensor, covariant of rank m and contravariant of rank l , where $Y_{u_1 \dots u_p}^{t_1 \dots t_q}$ is an arbitrary mixed tensor contravariant of rank q and covariant of rank p , then the $n^{(p+l+q+m)}$ functions $X_{b_1 \dots b_q s_1 \dots s_m}^{\alpha_1 \dots \alpha_p r_1 \dots r_l}$ form a mixed tensor, contravariant of rank $(p + l)$ and covariant of rank $(q + m)$.

THEOREM 1.† If Φ is any invariant function of a tensor $E_{s_1 \dots s_q}^{r_1 \dots r_p}$, contravariant of rank p , covariant of rank q and of any other tensors which are independent of $E_{s_1 \dots s_q}^{r_1 \dots r_p}$, then $\partial\Phi/\partial E_{s_1 \dots s_q}^{r_1 \dots r_p}$ is a mixed tensor which is contravariant of rank q and covariant of rank p .

Proof: Let

$$\partial\Phi/\partial E_{s_1 \dots s_q}^{r_1 \dots r_p} = X_{r_1 \dots r_p}^{s_1 \dots s_q}.$$

Then

$$\bar{X}_{r_1 \dots r_p}^{s_1 \dots s_q} = \partial\Phi/\partial \bar{E}_{s_1 \dots s_q}^{r_1 \dots r_p} = \partial\Phi/\partial E_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} \times \partial E_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} / \partial \bar{E}_{s_1 \dots s_q}^{r_1 \dots r_p}.$$

But, by hypothesis,

$$E_{\beta_1 \beta_2 \dots \beta_q}^{\alpha_1 \alpha_2 \dots \alpha_p} = \bar{E}_{\sigma_1 \dots \sigma_q}^{\rho_1 \dots \rho_p} \frac{\partial x^{\alpha_1}}{\partial \bar{x}^{\rho_1}} \dots \frac{\partial x^{\alpha_p}}{\partial \bar{x}^{\rho_p}} \frac{\partial \bar{x}^{\sigma_1}}{\partial x^{\beta_1}} \dots \frac{\partial \bar{x}^{\sigma_q}}{\partial x^{\beta_q}}.$$

* See F. D. Murnaghan, *Vector Analysis and Theory of Relativity*, p. 25.

† The author finds that this theorem is not essentially different from the "Aronhold Process." See Weitzenböck, *Invarianten Theorie*, p. 18.

Hence

$$\frac{\partial E_{\beta_1 \dots \beta_q}^{a_1 \dots a_p}}{\partial \bar{x}^{s_1 \dots s_q}} = \frac{\partial x^{a_1}}{\partial \bar{x}^{r_1}} \dots \frac{\partial x^{a_p}}{\partial \bar{x}^{r_p}} \frac{\partial \bar{x}^{s_1}}{\partial x^{\beta_1}} \dots \frac{\partial \bar{x}^{s_q}}{\partial x^{\beta_q}}.$$

Therefore

$$\bar{X}_{r_1 \dots r_p}^{s_1 \dots s_q} = X_{a_1 \dots a_p}^{\beta_1 \dots \beta_q} \frac{\partial x^{a_1}}{\partial \bar{x}^{r_1}} \dots \frac{\partial x^{a_p}}{\partial \bar{x}^{r_p}} \frac{\partial \bar{x}^{s_1}}{\partial x^{\beta_1}} \dots \frac{\partial \bar{x}^{s_q}}{\partial x^{\beta_q}}.$$

Q. E. D.

Now suppose we have a mixed tensor $X_{s_1 \dots s_q}^{r_1 \dots r_p}$ which is contravariant of rank p and covariant of rank q and is a function of a mixed tensor $E_{b_1 \dots b_m}^{a_1 \dots a_l}$ and possibly other tensors. Let us form an invariant Φ from the tensor $X_{s_1 \dots s_q}^{r_1 \dots r_p}$ and the arbitrary tensor $Y_{n_1 \dots n_p}^{m_1 \dots m_q}$, which is covariant of rank p and contravariant of rank q and which is independent of the tensor $E_{b_1 \dots b_m}^{a_1 \dots a_l}$. By the rule of composition, we have

$$\Phi = X_{\beta_1 \dots \beta_q}^{a_1 \dots a_p} \cdot Y_{a_1 \dots a_p}^{\beta_1 \dots \beta_q},$$

Now by Theorem 1

$$\frac{\partial \Phi}{\partial E_{b_1 \dots b_m}^{a_1 \dots a_l}} = Z_{a_1 \dots a_l}^{b_1 \dots b_m},$$

is a mixed tensor contravariant of rank m and covariant of rank l . But

$$\frac{\partial \Phi}{\partial E_{b_1 \dots b_m}^{a_1 \dots a_l}} = \left(\frac{\partial X_{\beta_1 \dots \beta_q}^{a_1 \dots a_p}}{\partial E_{b_1 \dots b_m}^{a_1 \dots a_l}} \right) Y_{a_1 \dots a_p}^{\beta_1 \dots \beta_q} = Z_{a_1 \dots a_l}^{b_1 \dots b_m}.$$

Hence, denoting

$$\frac{\partial X_{\beta_1 \dots \beta_q}^{a_1 \dots a_p}}{\partial E_{b_1 \dots b_m}^{a_1 \dots a_l}} \text{ by } F_{\beta_1 \dots \beta_q a_1 \dots a_l}^{a_1 \dots a_p b_1 \dots b_m},$$

we have

$$F_{\beta_1 \dots \beta_q a_1 \dots a_l}^{a_1 \dots a_p b_1 \dots b_m} Y_{a_1 \dots a_p}^{\beta_1 \dots \beta_q} = Z_{a_1 \dots a_l}^{b_1 \dots b_m},$$

and, therefore, by the converse of the rule of composition of tensors,

$$F_{s_1 \dots s_q a_1 \dots a_l}^{r_1 \dots r_p b_1 \dots b_m} = \frac{\partial X_{s_1 \dots s_q}^{r_1 \dots r_p}}{\partial E_{b_1 \dots b_m}^{a_1 \dots a_l}}$$

is a mixed tensor which is contravariant of rank $(p + m)$ and covariant of rank $(q + l)$. We have therefore proved

THEOREM 2. *If $X_{s_1 \dots s_q}^{r_1 \dots r_p}$ is a mixed tensor, contravariant of rank p and covariant of rank q which is a function of a mixed tensor $E_{b_1 \dots b_m}^{a_1 \dots a_l}$, and possibly other tensors, then $\frac{\partial X_{s_1 \dots s_q}^{r_1 \dots r_p}}{\partial E_{b_1 \dots b_m}^{a_1 \dots a_l}}$ is a mixed tensor which is covariant of rank $(q + l)$ and contravariant of rank $(p + m)$.*

2. We shall now apply the first of the above theorems to derive tensors from the invariants of the mixed tensor of rank 2, E_s^r , and shall consider the problem of expressing the invariants of these derived tensors in terms

of the invariants of E_s^r . Note: The invariants of a covariant tensor of rank 2, E_{rs} , are the invariants of the n -ary quadratic form $f = E_{\alpha\beta}x^\alpha x^\beta$ and there is only one invariant of this form, namely its discriminant.* But $E_s^r = g^{ra}E_{as}$ where $g^{rs} = \text{cofactor of } g_{sr} \div |g|$ and g_{rs} are the coefficients of the differential form $ds^2 = g_{\alpha\beta} \cdot dx^\alpha dx^\beta$, thus the invariants of E_s^r may be considered as the simultaneous invariants of the two n -ary quadratic forms $f = E_{\alpha\beta}x^\alpha x^\beta$ and $g = g_{\alpha\beta}x^\alpha x^\beta$. Or, if one prefers, the invariants of E_s^r may be regarded as the simultaneous invariants of E_s^r and the unit tensor δ_s^r whose components are 1 or zero according as r is equal to s or not.

Introducing the generalized Kronecker delta † $\delta_{s_1 \dots s_k}^{r_1 \dots r_k}$ (which, if the superscripts are distinct from each other and the subscripts are the same set of numbers as the superscripts, has the value +1 or -1 according as an even or an odd permutation is required to arrange the superscripts in the same order as the subscripts; and which in all other cases has the value zero) we can form the following invariants of the tensor E_s^r :

$$\begin{aligned} I_1 &= \delta_\alpha^\alpha E_\alpha^\alpha; & I_2 &= (1/2!) \delta_{\beta_1 \beta_2}^{\alpha_1 \alpha_2} E_{\alpha_1}^{\beta_1} E_{\alpha_2}^{\beta_2}, \\ I_3 &= (1/3!) \delta_{\alpha_1 \alpha_2 \alpha_3}^{\beta_1 \beta_2 \beta_3} E_{\alpha_1}^{\beta_1} E_{\alpha_2}^{\beta_2} E_{\alpha_3}^{\beta_3}; \\ &\dots \dots \dots \\ I_n &= (1/n!) \delta_{\beta_1 \beta_2 \dots \beta_n}^{\alpha_1 \alpha_2 \dots \alpha_n} E_{\alpha_1}^{\beta_1} E_{\alpha_2}^{\beta_2} \dots E_{\alpha_n}^{\beta_n}. \end{aligned}$$

Applying Theorem 1 to these invariants, we derive from each of them a mixed tensor. Denote the tensors derived from $I_1, I_2, \dots, I_n$ by $T_{1r}^s, T_{2r}^s, \dots, T_{nr}^s$ respectively. Then

$$\begin{aligned} T_{1r}^s &= \partial I_1 / \partial E_s^r = \delta_r^s, \\ T_{2r}^s &= \partial I_2 / \partial E_s^r = \delta_r^s E_\alpha^\beta - \delta_\beta^s \delta_r^\alpha E_\alpha^\beta = \delta_r^s I_1 - E_r^s, \\ T_{3r}^s &= \partial I_3 / \partial E_s^r = (1/2!) \delta_{\beta_1 \beta_2}^{\alpha_1 \alpha_2} E_{\alpha_1}^{\beta_1} E_{\alpha_2}^{\beta_2} \\ &= (1/2!) \{ \delta_r^s \delta_{\beta_1 \beta_2}^{\alpha_1 \alpha_2} - \delta_{\beta_1}^s \delta_{\beta_2}^{\alpha_1 \alpha_2} - \delta_{\beta_2}^s \delta_{\beta_1}^{\alpha_1 \alpha_2} \} E_{\alpha_1}^{\beta_1} E_{\alpha_2}^{\beta_2} \\ &= \delta_r^s I_2 - (1/2!) \{ T_{2r}^{\alpha_1} E_{\alpha_1}^s + T_{2r}^{\alpha_2} E_{\alpha_2}^s \} = \delta_r^s I_2 - T_{2r}^{\alpha} E_\alpha^s \\ &= \delta_r^s I_2 - (\delta_r^\alpha I_1 - E_r^\alpha) E_\alpha^s = \delta_r^s I_2 - I_1 E_r^s + E_r^\alpha E_\alpha^s, \\ T_{4r}^s &= (1/3!) \delta_{\beta_1 \beta_2 \beta_3}^{\alpha_1 \alpha_2 \alpha_3} E_{\alpha_1}^{\beta_1} E_{\alpha_2}^{\beta_2} E_{\alpha_3}^{\beta_3} \\ &= (1/3!) \{ \delta_r^s \delta_{\beta_1 \beta_2 \beta_3}^{\alpha_1 \alpha_2 \alpha_3} - (\delta_{\beta_1}^s \delta_{\beta_2 \beta_3}^{\alpha_1 \alpha_2 \alpha_3} + \delta_{\beta_2}^s \delta_{\beta_1 \beta_3}^{\alpha_1 \alpha_2 \alpha_3} + \delta_{\beta_3}^s \delta_{\beta_1 \beta_2}^{\alpha_1 \alpha_2 \alpha_3}) \} E_{\alpha_1}^{\beta_1} E_{\alpha_2}^{\beta_2} E_{\alpha_3}^{\beta_3} \\ &= \delta_r^s I_3 - (1/3) \{ T_{2r}^{\alpha_1} E_{\alpha_1}^s + T_{2r}^{\alpha_2} E_{\alpha_2}^s + T_{2r}^{\alpha_3} E_{\alpha_3}^s \} = \delta_r^s I_3 - T_{2r}^{\alpha} E_\alpha^s \\ &= \delta_r^s I_3 - I_2 E_r^s + I_1 E_r^\alpha E_\alpha^s - E_r^{\alpha_1} E_{\alpha_1}^{\alpha_2} E_{\alpha_2}^s, \\ &\dots \dots \dots \\ T_{pr}^s &= \delta_r^s I_{p-1} - I_{p-2} E_r^s + I_{p-3} E_r^\alpha E_\alpha^s - I_{p-4} E_r^{\alpha_1} E_{\alpha_1}^{\alpha_2} E_{\alpha_2}^s \\ &\quad + \dots (-1)^{t+1} I_{p-t} E_r^{\alpha_1} E_{\alpha_1}^{\alpha_2} \dots E_{\alpha_{t-2}}^s + \dots (-1)^{p+1} E_r^{\alpha_1} E_{\alpha_1}^{\alpha_2} \dots E_{\alpha_{p-2}}^s. \end{aligned}$$

* See Dickson, *Algebraic Invariants*, p. 48.

† See F. D. Murnaghan, *American Mathematical Monthly*, Vol. 32, p. 233.

where

$$f(\lambda) = \lambda^n - I_1 \lambda^{n-1} + I_2 \lambda^{n-2} + \cdots (-1)^n I_n = |\delta_r^s \lambda - E_r^s|.$$

By Taylor's expansion for a polynomial of degree n ,

$$f(I_1 - \lambda) = f(I_1) - \lambda f'(I_1) + (\lambda^2/2!) f''(I_1) + \cdots (-1)^n \lambda^n;$$

so that the characteristic polynomial for T_{2r}^s is

$$\lambda^n - [f^{(n-1)}(I_1)/(n-1)!] \lambda^{n-1} + [f^{(n-2)}(I_1)/(n-2)!] \lambda^{n-2} + \cdots (-1)^r [f^{(n-r)}(I_1)/(n-r)!] \lambda^{n-r} + \cdots (-1)^n f(I_1),$$

from which we obtain the invariants of T_{2r}^s . They are

$$\begin{aligned} I_{12} &= [f^{(n-1)}(I_1)/(n-1)!] = 1/(n-1)! \{n! I_1 - (n-1)! I_1\} = (n-1) I_1, \\ I_{22} &= [f^{(n-2)}(I_1)/(n-2)!] \\ &= 1/(n-2)! \{ (n!/2!) I_1^2 - (n-1)! I_1 I_1 + (n-2)! I_2 \} \\ &= [n(n-1)/2!] I_1^2 - (n-1) I_1 I_1 + I_2, \\ I_{32} &= [f^{(n-3)}(I_1)/(n-3)!] \\ &= 1/(n-3)! \{ (n!/3!) I_1^3 - [(n-1)!/2!] I_1 I_1^2 \\ &\quad + [(n-2)!/1!] I_2 I_1 - (n-3)! I_3 \} = [n(n-1)(n-2)/3!] I_1^3 \\ &\quad - [(n-1)(n-2)/2!] I_1 I_1^2 - (n-2) I_2 I_1 - I_3, \\ I_{s2} &= [f^{(n-s)}(I_1)/(n-s)!] = 1/(n-s)! \{ (n!/s!) I_1^s \\ &\quad - [(n-1)!/(s-1)!] I_1 I_1^{s-1} + [(n-2)!/(s-2)!] I_2 I_1^{s-2} \\ &\quad + \cdots (-1)^r [(n-r)!/(s-r)!] I_r I_1^{s-r} + \cdots (-1)^s (n-s)! I_s \} \\ &= [n!/(n-s)! s!] I_1^s - [(n-1)!/(n-s)! (s-1)!] I_1 I_1^{s-1} \\ &\quad + [(n-2)!/(n-s)! (s-2)!] I_2 I_1^{s-2} \\ &\quad + \cdots (-1)^r [(n-r)!/(n-s)! (s-r)!] I_r I_1^{s-r} + \cdots (-1)^s I_s, \\ I_{n2} &= f(I_1) = I_1^n - I_1 I_1^{n-1} + I_2 I_1^{n-2} + \cdots (-1)^r I_r I_1^{n-r} + \cdots (-1)^n I_n. \end{aligned}$$

Likewise the invariants $I_{13}, I_{23}, \cdots I_{n3}$ of the tensor T_{3r}^s will be the coefficients of the characteristic equation of the matrix $\|T_{3r}^s\|$ and since (from the expression for T_{3r}^s on page 198)

$$\|T_{3r}^s\| = \|\delta_r^s I_2\| - I_1 \|E_r^s\| + \|E_r^s\|^2$$

by the theorem stated above, $I_{13}, I_{23}, \cdots I_{n3}$ will be the coefficients of the equation whose roots are $(I_2 - I_1 \lambda_1 + \lambda_1^2), (I_2 - I_1 \lambda_2 + \lambda_2^2), \cdots (I_2 - I_1 \lambda_n + \lambda_n^2)$ where $\lambda_1, \lambda_2, \cdots \lambda_n$ are the roots of

$$f(\lambda) = \lambda^n - I_1 \lambda^{n-1} + I_2 \lambda^{n-2} + \cdots (-1)^n I_n = 0.$$

From the elementary theory of equations we see that for this last equation

$$\begin{aligned} I_1 &= \Sigma \lambda_1 \\ I_2 &= \Sigma \lambda_1 \lambda_2 \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ I_n &= \Sigma \lambda_1 \lambda_2 \cdots \lambda_n. \end{aligned}$$

Similarly, since $I_{13}, I_{23}, \cdots I_{n3}$ play a corresponding rôle in the characteristic equation of the matrix $\|T_{3r^s}\|$, we shall have

$$\begin{aligned} I_{13} &= \Sigma(I_2 - I_1\lambda_1 + \lambda_1^2), \\ I_{23} &= \Sigma(I_2 - I_1\lambda_1 + \lambda_1^2)(I_2 - I_1\lambda_2 + \lambda_2^2), \\ I_{33} &= \Sigma(I_2 - I_1\lambda_1 + \lambda_1^2)(I_2 - I_1\lambda_2 + \lambda_2^2)(I_2 - I_1\lambda_3 + \lambda_3^2), \text{ etc.} \end{aligned}$$

By multiplying out the factors of a term in the above summations we find

$$\begin{aligned}
I_{13} &= nI_2 - I_1\Sigma\lambda_1 + \Sigma\lambda_1^2, \\
I_{23} &= [n(n-1)/2!]I_2^2 - (n-1)I_2I_1\Sigma\lambda_1 \\
&\quad + (n-1)I_2\Sigma\lambda_1^2 + I_1^2\Sigma\lambda_1\lambda_2 - I_1\Sigma\lambda_1\lambda_2^2 + \Sigma\lambda_1^2\lambda_2^2 \\
&= [n(n-1)/2!]I_2^2 - (n-1)I_2\{I_1\Sigma\lambda_1 - \Sigma\lambda_1^2\} + \{\Sigma(I_1\lambda_1 - \lambda_1^2)(I_1\lambda_2 - \lambda_2^2)\}, \\
I_{33} &= [n(n-1)(n-2)/3!]I_2^3 - [(n-1)(n-2)/2!]I_2^2I_1\Sigma\lambda_1 \\
&\quad + [(n-1)(n-2)/2!]I_2^2\Sigma\lambda_1^2 + (n-2)I_2I_1^2\Sigma\lambda_1\lambda_2 \\
&\quad - (n-2)I_2I_1\Sigma\lambda_1\lambda_2^2 + (n-2)I_2\Sigma\lambda_1^2\lambda_2^2 \\
&\quad + \Sigma\lambda_1^2\lambda_2^2\lambda_3^2 - I_1\Sigma\lambda_1\lambda_2^2\lambda_3^2 + I_1^2\Sigma\lambda_1\lambda_2\lambda_3^2 - I_1^3\Sigma\lambda_1\lambda_2\lambda_3 \\
&= [n(n-1)(n-2)/3!]I_2^3 - [(n-1)(n-2)/2!]I_2^2\{I_1\Sigma\lambda_1 - \Sigma\lambda_1^2\} \\
&\quad + (n-2)I_2\{\Sigma(I_1\lambda_1 - \lambda_1^2)(I_1\lambda_2 - \lambda_2^2)\} \\
&\quad - \{\Sigma(I_1\lambda_1 - \lambda_1^2)(I_1\lambda_2 - \lambda_2^2)(I_1\lambda_3 - \lambda_3^2)\}.
\end{aligned}$$

From which we see that

$$I_{rs} = [n!/(n-r)!r!] I_2^r - [(n-1)!/(n-r)!(r-1)!] I_2^{r-1} \{I_1 \Sigma \lambda_1 - \Sigma \lambda_1^2\} \\ + \cdots (-1)^{s-1} [(n-s+1)!/(n-r)!(r-s+1)!] I_2^{r-s+1} \\ \times \{\Sigma(I_1 \lambda_1 - \lambda_1^2) \cdots (I_1 \lambda_{s-1} - \lambda_{s-1}^2)\} \\ + \cdots (-1)^r \{\Sigma(I_1 \lambda_1 - \lambda_1^2) \cdots (I_1 \lambda_r - \lambda_r^2)\}.$$

$$\begin{aligned} \text{Substituting } I_1 &= \Sigma \lambda_1 \text{ and } \Sigma \lambda_1^2 = I_1^2 - 2I_2, \\ I_{13} &= nI_2 - I_1^2 + I_1^2 - 2I_2 = (n-2)I_2. \end{aligned}$$

Furthermore it is easily seen that

$$\Sigma \lambda_1 \lambda_2^2 = I_1 I_2 - 3I_3, \quad \Sigma \lambda_1^2 \lambda_2^2 = I_2^2 - 2I_1 I_3 + 2I_4, \quad \Sigma \lambda_1 \lambda_2 = I_2.$$

Substituting these results in the expression (page 201) for I_{23} we obtain

$$I_{23} = [n(n-1)/2!]I_2^2 - 2(n-1)I_2^2 + I_2^2 + I_1I_3 + 2I_4.$$

Since $\Sigma \lambda_1^2 \lambda_2^2 \lambda_3^2$ is of degree 6 in the λ 's, it is of weight six in the coefficients I , and we write

$$\Sigma \lambda_1^2 \lambda_2^2 \lambda_3^2 = AI_2I_5 + BI_2I_4 + CI_3^2 + DI_6.$$

Let

$$\lambda_1 = \lambda_2 = \lambda_3 = 1; \quad \lambda_4 = \lambda_5 = \dots = \lambda_n = 0.$$

Then

$$I_1 = 3, \quad I_2 = 3, \quad I = 1, \quad I_4 = I_5 = 0.$$

Substituting we obtain $C = 1$.

Next let $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1; \lambda_5 = \dots = \lambda_n = 0$. Then

$$I_1 = 4, \quad I_2 = 6, \quad I_3 = 4, \quad I_4 = 1, \quad I_5 = I_6 = 0.$$

We get $B = -2$.

Continuing in this way we find that

$$\Sigma \lambda_1^2 \lambda_2^2 \lambda_3^2 = 2I_1I_5 - 2I_2I_4 + I_3^2 - 2I_6.$$

And by a similar process we obtain

$$\Sigma \lambda_1 \lambda_2^2 \lambda_3^2 = I_2I_3 - 3I_1I_4 + 5I_5, \quad \Sigma \lambda_1 \lambda_2 \lambda_3^2 = I_1I_3 - 4I_4, \quad \Sigma \lambda_1 \lambda_2 \lambda_3 = I_3.$$

Substituting these and the foregoing results in the expression (page 13) for I_{33} , we find

$$\begin{aligned} I_{33} = & [n(n-1)(n-2)/3!]I_2^3 - [(n-1)(n-2)/2!]I_2^2(2I_2) \\ & + (n-2)I_2(I_2^2 + I_1I_3 + 2I_4) \\ & - (2I_6 + I_1I_2I_3 + I_1^2I_4 + 3I_1I_5 + 2I_2I_4 + I_3^2). \end{aligned}$$

We observe from the expression for I_{r3} on page 201 that this invariant is a polynomial of degree r in I_2 and that the coefficients of this polynomial are symmetric functions of the roots of $f(\lambda) = 0$. These symmetric functions being of the form $\Sigma \lambda_1^p \lambda_2^q \lambda_3^r \dots \lambda_s^t$ where the exponents are at most 2, can be calculated by means of known* formulae in terms of the coefficients of $f(\lambda) = 0$, i. e. in terms of the invariants $I_1, I_2, \dots, I_n$. It is evident from the foregoing expressions for the I_{13}, I_{23}, I_{33} , however, that the expression for I_{r3} is not conveniently put explicitly in terms of I_1, I_2 , etc.

Finally we observe from the expressions (page 198) for the tensors T_r^s that each of the matrices $\|T_r^s\|$ is a polynomial function of the matrix $\|E_r^s\|$.

* See Burnside and Panton, *Theory of Equations*, p. 269.

Thus

$$\begin{aligned}\|T_{2r}^s\| &= \|\delta_r^s I_1\| - \|E_r^s\|, \\ \|T_{3r}^s\| &= \|\delta_r^s I_2\| - I_1 \|E_r^s\| + \|E_r^s\|^2, \\ \|T_{4r}^s\| &= \|\delta_r^s I_3\| - I_2 \|E_r^s\| + I_1 \|E_r^s\|^2 - \|E_r^s\|^3, \text{ etc.}\end{aligned}$$

It follows from the theorem stated on page 199 that the latent roots of each of these matrices are the corresponding polynomial functions of the roots of $f(\lambda)=0$. Now given an equation of degree n , $f(\lambda)=0$ whose roots are $\lambda_1, \lambda_2 \cdots \lambda_n$ we can, at least theoretically, by means of Tschirnhausen's transformation,* form the equation whose roots are any polynomial function of degree $(n-1)$ or less of the roots of $f(\lambda)=0$, and the coefficients of the transformed equation will be given in terms of the coefficients of $f(\lambda)=0$. Thus the invariants (coefficients of the characteristic equation of the matrix $\|T_{pr}^s\|$) of any one of the tensors T_{pr}^s are functions of the invariants (coefficients of the equation $f(\lambda)=0$) of the tensor E_r^s .

In general we can say, as is evident from the foregoing considerations, if F_s^r is any polynomial function

$$F = a_0 + a_1 E + a_2 E^2 + \cdots,$$

of the matrix E then its invariants can be expressed in terms of the invariants of E , and hence the derivatives, with respect to E_s^r , of its invariants can be expressed in terms of the tensors $T_{1r}^s, T_{2r}^s \cdots T_{nr}^s$.

* See Burnside and Panton, *op. cit.*, p. 318.

VITA.

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